

BLACK HOLE SOLUTIONS WITH DILATONIC HAIR IN HIGHER CURVATURE GRAVITY

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A new numerical integration method for examining a black hole structure was realized. Black hole solutions with dilatonic hair of 4D low energy effective SuperString Theory action with Gauss-Bonnet quadratic curvature contribution were studied, using this method, inside and outside the event horizon. Thermodynamical properties of this solution were also studied.

0260Lj, 0270Rw, 0425Dm, 0470Bw, 0470Dy

I. INTRODUCTION

During last years a great interest for an investigation of low energy effective SuperString Theory action in four dimensions arised. Some researches [1–4] found that the well-known solutions (such as Schwarzschild one and ets.) were modified by the higher order curvature corrections. They also showed that the “no-hair” theorem can not been applied to those modified configurations.

The problem is to find black-hole solutions of 4D low energy String effective action with the second order curvature corrections. For simplicity, as a rule, researchers [2,3] consider the bosonic part of the gravitational action consisting of dilaton, graviton and Gauss-Bonnet (GB) terms taken in the following form:

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[m_{pl}^2 (-R + 2\partial_\mu \phi \partial^\mu \phi) + \lambda e^{-2\phi} S_{GB} \right], \quad (1)$$

where R is a scalar curvature; ϕ is a dilaton field; m_{pl} is the Plank mass; λ is the string coupling parameter. The later describes GB contribution ($S_{GB} = R_{ijkl}R^{ijkl} - 4R_{ij}R^{ij} + R^2$) to the action (1). Such configurations were partly studied [2] by the perturbative analysis $O(\lambda)$ outside the event horizon when event horizon radius $r_h \gg m_{pl}$. Authors showed the black hole solution to be real and it provides the non-trivial dilatonic hair. P.Kanti et al [3] obtained the similar solution using the non-perturbative numerical method outside the horizon. However, a question on a solution behavior inside the event horizon is still open. It is generally believed [5] that in the regions where the spacetime curvature is just small a classical solution gives the main contribution to the global structure of the spacetime. Quantum corrections may drastically modify the spacetime properties in the case of large enough curvature. The purpose of this paper is to study the complete black-hole solution.

Structure of our paper is the following: in Section 2 an analytical investigation of the action (1) is described; Section 3 deals with the numerical results; Section 4 is devoted to description of thermodynamical properties of the solution; Section 5 contains a discussion and conclusions. In Appendix one can find a brief description of our integration method.

II. ANALYTICAL INVESTIGATION OF THE GAUSS-BONNET ACTION

The aim is to find static, asymptotically flat, spherically symmetric black-hole-like solutions. In this case the most convenient choice of metric is the following:

$$ds^2 = \Delta dt^2 - \frac{\sigma^2}{\Delta} dr^2 - f^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (2)$$

where functions Δ , σ and f depend only on radial coordinate r . Therefore, the scalar curvature R and the GB term S_{GB} have the following forms:

$$R\sqrt{-g}\frac{1}{\sin\theta} = \left[\frac{\Delta' f^2}{\sigma} + 4\frac{f'\Delta f}{\sigma} \right]' - \frac{2}{\sigma} \left[\Delta' f' f + \Delta(f')^2 + \sigma^2 \right], \quad (3)$$

$$S_{GB}\sqrt{-g}\frac{1}{\sin\theta} = 4 \left[\frac{\Delta\Delta'(f')^2}{\sigma^3} - \frac{\Delta'}{\sigma} \right]'. \quad (4)$$

Integrating i) by parts the GB term and R term and ii) over the angle variables in (1), one can rewrite the action in somewhat more convenient form (for the present analysis the boundary term is not relevant and is ignored):

$$S = \frac{1}{2} \int dt dr \left[m_{pl}^2 \frac{1}{\sigma} (\Delta' f' f + \Delta(f')^2 + \sigma^2 - \Delta f^2 (\phi')^2) + 4e^{-2\phi} \lambda \phi' \left(\frac{\Delta\Delta'(f')^2}{\sigma^3} - \frac{\Delta'}{\sigma} \right) \right]. \quad (5)$$

Corresponding field equations in the curvature gauge ($f(r) = r$) are the following:

$$m_{pl}^2 \sigma^2 [-r\sigma' + \sigma r^2 (\phi')^2] + 4e^{-2\phi} \lambda \sigma (\Delta - \sigma^2) [\phi'' - 2(\phi')^2] + 4e^{-2\phi} \phi' \lambda \sigma' (\sigma^2 - 3\Delta) = 0, \quad (6)$$

$$m_{pl}^2 \sigma^2 [\sigma^2 + \Delta r^2 (\phi')^2 - \Delta' r - \Delta] + 4e^{-2\phi} \phi' \lambda \Delta' (\sigma^2 - 3\Delta) = 0, \quad (7)$$

$$m_{pl}^2 \sigma^2 [\Delta'' r \sigma - \Delta' r \sigma' + 2\Delta' \sigma - 2\Delta \sigma' + 2\Delta r \sigma (\phi')^2] + 4e^{-2\phi} \lambda 2\sigma \Delta \Delta' [\phi'' - 2(\phi')^2] + 4e^{-2\phi} \phi' \lambda 2[(\Delta')^2 \sigma + \Delta \Delta'' \sigma - 3\Delta \Delta' \sigma'] = 0, \quad (8)$$

$$-2m_{pl}^2 \sigma^2 [\Delta' r^2 \sigma \phi' + 2\Delta r \phi' \sigma - \Delta r^2 \phi' \sigma' + \Delta r^2 \phi'' \sigma] + 4e^{-2\phi} \lambda [(\Delta')^2 \sigma + \Delta \Delta'' \sigma - 3\Delta \Delta' \sigma' - \Delta'' \sigma^3 + \Delta' \sigma' \sigma^2] = 0. \quad (9)$$

The equation (7) can be solved analytically and it permits to get the following expression for σ :

$$\sigma^2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad (10)$$

where

$$a = m_{pl}^2, \quad b = m_{pl}^2 (\Delta r^2 (\phi')^2 - \Delta' r - \Delta) + 4e^{-2\phi} \lambda \phi' \Delta', \quad c = -4e^{-2\phi} \lambda \phi' (3\Delta \Delta').$$

In a formal mathematical way it is necessary to put the formula (10) to equations (6), (8), (9) and to work with two unknown functions Δ and ϕ , but in the given situation it is more convenient to work with the equations (6), (8), (9) and three unknown functions Δ , σ and ϕ , verifying (7) at every numerical integration step and writing asymptotic expressions for all four equations. We choose the later strategy.

Let us suppose the solutions behaviors near the event horizon to have the following forms:

$$\Delta = d_1 x + d_2 x^2 + O(x^2), \quad (11)$$

$$\sigma = s_0 + s_1 x + O(x), \quad (12)$$

$$e^{-2\phi} = \phi_0 (1 - 2 * \phi_1 x + 2(\phi_1^2 - \phi_2) x^2) + O(x^2), \quad (13)$$

where $x = r - r_h \ll 1$. Putting formula (11)-(13) in the equations (6)-(9), we obtain the following equation and relations between the expansion coefficients (s_0 , s_1 and ϕ_0 are free parameters):

$$z_1 d_1^2 + z_2 d_1 + z_3 = 0, \quad (14)$$

where:

$$z_1 = 24\lambda^2 \phi_0^2 r_h, \quad z_2 = 8\lambda^2 \phi_0^2 s_0^2 - m_{pl}^4 r_h^4 s_0^2, \quad z_3 = m_{pl}^4 r_h^3 s_0^4,$$

and other parameters are:

$$d_2 = \left(-\frac{1}{2r_h s_0^2} \right) [r_h d_1^2 + d_1 s_0^2 - r_h s_0 s_1 d_1], \quad (15)$$

$$\phi_1 = \frac{m_{pl}^2}{4\lambda d_1 \phi_0} [r_h d_1 - s_0^2], \quad (16)$$

$$\phi_2 = \frac{1}{8\lambda s_0 \phi_0} [8\lambda \phi_1^2 \phi_0 + 4\lambda s_1 \phi_1 \phi_0 + r_h^2 s_0 \phi_1^2 m_{pl}^2 - r_h s_1 m_{pl}^2]. \quad (17)$$

The solution of quadratic equation (14) exists if a value of the free parameter ϕ_0 satisfies the following condition:

$$\phi_0^2 \geq \frac{7(1 + \sqrt{195})m_{pl}^4 r_h^4}{2\lambda^2}. \quad (18)$$

This is a natural restriction to the solution existence of the equation system (6)-(9).

We are interesting in asymptotically flat solutions. Therefore, its infinity behavior is to be as follows:

$$\Delta = 1 - \frac{2M}{r} + O\left(\frac{1}{r}\right); \quad \sigma = 1 + O\left(\frac{1}{r}\right); \quad \phi = \phi_\infty + \frac{D}{r} + O\left(\frac{1}{r}\right), \quad (19)$$

where ADM mass M and dilaton charge D can be calculated from the equations (8), (9) by the following formulae:

$$\begin{aligned} \left(\frac{\Delta' r^2}{\sigma}\right) = 2M = \int_{r_h}^{\infty} dr \frac{1}{\sigma^4 m_{pl}^2} & \left(2m_{pl}^2 \sigma^2 \left[\Delta r \sigma' - \Delta r^2 (\phi')^2 \sigma \right] \right. \\ & \left. - 4e^{-2\phi} \lambda 2\sigma \Delta \Delta' r [\phi'' - 2(\phi')^2] - 4e^{-2\phi} \phi' \lambda r [(\Delta')^2 \sigma + \Delta \Delta'' \sigma - 3\Delta \Delta' \sigma'] \right), \end{aligned} \quad (20)$$

$$\left(\frac{2\Delta r^2 \phi'}{\sigma}\right) = D = \int_{r_h}^{\infty} dr \frac{1}{m_{pl}^2 \sigma^4} \left(4e^{-2\phi} \lambda [(\Delta')^2 \sigma + \Delta \Delta'' \sigma - 3\Delta \Delta' \sigma' - \Delta'' \sigma^3 + \Delta' \sigma' \sigma^2] \right). \quad (21)$$

The dependencies of ADM mass M and dilaton charge D versus the event horizon radius r_h are shown in Tab.1.

III. NUMERICAL RESULTS

For integrating inside the event horizon a method based on integrating over an additional parameter was used. It is briefly described in Appendix. Here we present the main results.

The integration was done from the event horizon to the infinity, then, using data obtained, from the infinity to the horizon and, further, inside the horizon. The results of our calculation are shown in Fig.1. It presents the dependence of metric functions Δ and σ and the dilaton function ϕ versus r . Parameter r_h was allowed to change within wide range from 3.5 to 100.0 Plank unit values. Fig.2 represents 3D plot of the dependence of the metric function $\Delta(r)$ and dilaton function $\exp(-2\phi(r))$ versus r_h . As one can see from Fig.1 and Fig.2 the behaviors of Δ , σ and ϕ outside the horizon have the usual forms, as obtained in [2,3], and look like the standard Schwarzschild solution (ADM mass values M ($r_h \simeq 2M$) illustrate this, see Table 1). Under the horizon $r < r_h$ solution exist only till the value $r = r_s$. Another solution branch begins from the value r_s , but it exists only till the “singular” horizon r_x . The asymptotic behavior of both solution branches near the position r_s can be described by the following formulae, using the only smooth function σ as an independent variable:

$$\begin{aligned} \Delta &= d_s + d_2 x^2 + O(x^2); \\ r &= r_s + r_2 x^2 + O(x^2); \\ \exp(-2\phi) &= \phi_s (1 - 2f_2 x^2) + O(x^2), \end{aligned} \quad (22)$$

where $x = \sigma - \sigma_s \ll 1$. Free parameters are the following: σ_s , ϕ_s , r_s . Other expansion coefficients (d_s , f_2 , d_2/r_2) can be calculated from the following three equations:

$$d_2 = f_2, \quad (23)$$

$$m_{pl}^2 \sigma_s^2 \left[\sigma_s^2 + d_s r_s^2 \left(\frac{d_2}{r_2} \right)^2 - r_s \frac{d_2}{r_2} - d_s \right] + 4\phi_s \lambda \left(\frac{d_2}{r_2} \right)^2 (\sigma_s^2 - 3d_s) = 0, \quad (24)$$

$$\begin{aligned} & m_{pl}^6 \sigma_s^6 d_s r_s^4 + 4\phi_s \lambda m_{pl}^4 \frac{d_2}{r_2} \sigma_s^4 d_s^2 r_s^3 \\ & + (4\phi_s \lambda)^2 m_{pl}^2 \sigma_s^2 \left[4 \left(\frac{d_2}{r_2} \right)^2 d_s^3 r_s^2 + (d_s + r_s \frac{d_2}{r_2}) (d_s - \sigma_s^2)^2 \right] + (4\phi_s \lambda)^3 3d_s \left(\frac{d_2}{r_2} \right)^2 (d_s - \sigma_s^2)^2 = 0. \end{aligned} \quad (25)$$

The equation (24) represents the asymptotic form of the equation (7). The formula (23) represents the asymptotic form of the equation (8) and the rest of equations (6), (9) reduce into an identity. The equation (25) is the consequence of the system (35) (see Appendix) because according to the existence theorem (see Appendix) the system (35) has a single solution only in the case of its main discriminant to be not equal to zero. In the case the zero main discriminant in some point of the solution trajectory, the uniqueness of the solution (35) will be violated. So, equation (25) is just that condition in the r_s , where two solutions exist. “Curvature invariant” $R_{ijkl}R^{ijkl}$ in our metric parametrization is equal to:

$$\begin{aligned} R_{ijkl}R^{ijkl} &= 4\frac{\Delta^2}{\sigma^4 r^4} + 8\frac{\Delta^2(\sigma')^2}{\sigma^6 r^2} - 8\frac{\Delta}{\sigma^2 r^4} - 8\frac{\Delta\Delta'\sigma'}{\sigma^5 r^2} + \frac{(\Delta'')^2}{\sigma^4} + 4\frac{(\Delta')^2}{\sigma^4 r^2} - 2\frac{\Delta''\Delta'\sigma'}{\sigma^5} + \frac{(\Delta')^2(\sigma')^2}{\sigma^6} + \frac{4}{r^4} \\ &= \frac{1}{x^2} \left(\frac{4d_s}{r_2 r_s^2 \sigma_s^6} + \frac{d_2}{2r_2^2 \sigma_s^6} \right) + O\left(\frac{1}{x}\right) \rightarrow \infty. \end{aligned} \quad (26)$$

According to the classification from [8], r_s represents pure scalar singularity. It is necessary to note, as we tested, that the weak energy condition, dominant one and strong one are realized at the value $r = r_s$ [6].

The most interesting result is the existence of the second solution branch within the “singular” horizon r_x . The position of r_x is situated inside the main horizon r_h . The asymptotic behavior of Δ , σ and ϕ near r_x are the following:

$$\Delta = d_1 x + d_2 x^{\frac{3}{2}} + O(x^{\frac{3}{2}}), \quad \sigma = \sigma_0 + \sigma_1 x^{\frac{1}{2}} + O(x^{\frac{1}{2}}), \quad \phi = \phi_x + \phi_1 x + \phi_2 x^{\frac{3}{2}} + O(x^{\frac{3}{2}}),$$

where $x = r_x - r \ll 1$, and free parameters are σ_0 , σ_1 , $\phi_0 = \exp(-2\phi_x)$, d_1 and r_x . Other coefficients can be found from the following conditions:

$$\phi_1 = \frac{m_{pl}^2 [d_1(1+r_x) - \sigma_0^2]}{4\phi_0 \lambda d_1}, \quad \phi_2 = \frac{-m_{pl}^2 r_x \sigma_1 + 4\phi_0 \lambda \phi_1 \sigma_1}{6\phi_0 \lambda \sigma_0}, \quad d_2 = \frac{2}{3} \frac{d_1 \sigma_1}{\sigma_0}.$$

“Curvature invariant” $R_{ijkl}R^{ijkl} \sim \frac{1}{x} + O(x^{\frac{1}{2}}) \rightarrow \infty$ and this means that r_x represents the “singular horizon”. The distance between r_x and r_h is rather long for the big values of r_h and decreases with decreasing r_h . In the limit point, defined by the equation (14), all points pour together $r_h = r_s = r_x$ and the solutions inside r_h do not exist. This can be seen from Fig.2.

IV. THERMODYNAMICAL PROPERTIES OF GAUSS-BONNET SOLUTION

If one works in Lorentzian spacetime, he can say nothing about thermodynamical properties of a solution obtained. Using the Euclidean version of the metric:

$$ds^2 = \Delta d\tau^2 + \frac{\sigma^2}{\Delta} dr^2 + r^2 d\Omega,$$

where τ is a periodic coordinate which range is from 0 to π [7–9], one can easily write the inverse temperature [11]:

$$\beta = 4\pi \sqrt{g_{00}g_{11}} \left[\frac{d}{dr} g_{00} \right]^{-1} \Big|_{r=r_h}. \quad (27)$$

From equations (11)-(13) it is possible to find that $\beta = 4\pi(s_0/d_1)$.

In the Euclidean frame the full action S_E after integrating over periodic time τ takes the following form:

$$\begin{aligned} S_E &= \frac{1}{4}\beta \int dr \left[m_{pl}^2 \left(-\frac{\Delta'' r^2}{\sigma} - 4\frac{\Delta' r}{\sigma} + \frac{\Delta' \sigma' r^2}{\sigma^2} - 2\frac{\Delta}{\sigma} + 4\frac{\Delta \sigma' r}{\sigma^2} + 2\sigma - 2(\phi')^2 \frac{\Delta r^2}{\sigma} \right) \right. \\ &\quad \left. + 4e^{-2\phi} \lambda \left[\frac{\Delta \Delta'}{\sigma^3} - \frac{\Delta'}{\sigma} \right]' \right]. \end{aligned} \quad (28)$$

The boundary term obtained as it was done in [7] has the following form:

$$K = \frac{1}{4}\beta \left[m_{pl}^2 \left(\frac{\Delta' r^2}{\sigma} + 2\frac{\Delta r}{\sigma} \right) + 4e^{-2\phi} \lambda \left(\frac{\Delta \Delta'}{\sigma^3} - \frac{\Delta'}{\sigma} \right) \right]. \quad (29)$$

For a non-extreme black-hole the boundaries of the spacetime manifold are set by the extreme values of the radius coordinate. They are $r = r_h$ and $r = \infty$. After removal of the conical singularity, the spacetime has the only boundary at $r = \infty$, because the black hole horizon is not a spacetime border [9]. So, we have to take into account only r_h value of boundary term K . It takes the following form:

$$K - K_0 = (K - K_0)_{r_h} = \frac{1}{4}\beta \left[m_{pl}^2 \left(\frac{d_1 r_h^2}{s_0} - 2r_h \right) - 4\phi_0 \lambda \frac{d_1}{s_0} \right] \quad (30)$$

in terms of the asymptotic solution on the event horizon (14). K_0 is the boundary contribution of the flat spacetime. In this case the Euclidean action I_E takes the following form:

$$I_E = \int_{r_h}^{\infty} dr S_E - (K - K_0)_{r_h}, \quad (31)$$

where S_E is defined by (29).

The physical entropy can be found by the following expression [2]:

$$S(r_h) = \beta \frac{\partial I}{\partial \beta} - I_E = \beta(r_h) \left(\frac{\partial I(r_h)}{\partial r_h} \right) \left(\frac{\partial \beta(r_h)}{\partial r_h} \right)^{-1} - I_E(r_h). \quad (32)$$

Plots for the GB and Schwarzschild entropy are shown in Fig.3.

V. DISCUSSION AND CONCLUSIONS

In this paper there are suggested the black hole solutions with non-trivial dilaton “hair” of low energy effective SuperString Theory with second order curvature corrections which are obtained independently from Kanti et al [3] and were found outside and inside the event horizon by the numerical method described in Appendix. The solutions are characterized by ADM mass M , dilaton charge D and asymptotic dilaton value ϕ_∞ . They are stable under the fluctuations of initial conditions. So, we can arrive to a conclusion that the SuperString Theory provides such situations by itself as can not be covered by the “no-hair” theorem [3]. As these solutions have the non-perturbative nature, as they have no limits with the perturbative parameter values.

The most interesting result of our work is the existence of r_s -singularity inside the black hole after adding the GB term to the action. This singularity has the topology $S^2 \times R^1$, i.e. it is an infinite (in time direction) “tube” of radius r_s . Similar “tube” in the Schwarzschild metric with an additional condition of $R^{abcd}R_{abcd}$ finiteness was discussed by V.Frolov et al. [5]. There are two solutions on this “tube”. The asymptotically flat solution, which is the main one, starts from r_s and continues without limits up to the infinity. In the case of r_h to be quite large and $r_h \gg r_s$ it is possible to suppose r_s to be approximately vanished, then the main solution looks like the standard Schwarzschild one with a constant dilatonic field (see Fig.2) which agrees with results of Mignemi [2] and Kanti [3]. The additional solution branch provides the existence of the “singular” inner horizon with $R_{ijkl}R^{ijkl} \rightarrow \infty$. Some solutions exist inside “tube” r_s , but they are unstable under the initial condition fluctuations, and we can not distinguish which branch — the main one or the additional one — they correspond to. One can suppose that this r_s singularity is the consequence of the bad metric parametrization choice. In order to prove (or not) this artifact, the solution in another metric was studied. This is the metric of such class that was suggested by P.Breitenlohner et al. [12] and has the following form:

$$ds^2 = \beta^2(\tau)dt^2 - d\tau^2 - r^2(\tau)d\Omega^2. \quad (33)$$

Topology $S^2 \times R^1$ inside the black hole horizon also exists in this metric. So the presence of r_s “tube” is not the consequence of the metric parametrization choice. The question on the more detail structure of the “tube” r_s is the subject of the next study.

The Gauss-Bonnet term influence to the solution behavior was also considered. When the horizon radius r_h is quite large or λ becomes quite small the main contribution to the action comes from the Einstein part ($r_h \gg r_s \Rightarrow r_s \simeq 0$ in comparison with r_h). While decreasing r_h (increasing λ), the GB influence increases, the dilaton charge absolute value becomes larger, the distance between r_s and r_h becomes smaller. Once the limit position is achieved where $r_h = r_s = r_x \equiv r_{hxs}$. This is a minimal point which provides the solution like the black hole type. In the point r_{hxs} the solution (main branch) exists only outside the event horizon.

The Euclidean action and the entropy behavior of Einstein–dilaton–GB solution was also examined. It was found that the inverse temperature is not far from the corresponding Schwarzschild one. When r_h becomes quite large, the entropy S is quite equal to the Schwarzschild one S_{SW} . During decreasing r_h , the difference between S and S_{SW} becomes larger. S has always the positive signature. This conclusion does not contradict to Mignemi's formula [2], obtained by the perturbative method in the case of rather large r_h , which has the following form in our variables:

$$S = 4\pi M^2 \left(1 + \lambda \frac{1}{M^2} + \lambda^2 \frac{73}{120} \frac{1}{M^4} \right). \quad (34)$$

Our entropy value (32) was obtained with the non-perturbative method. So, it is possible to note, that the formula (34) is indeed correct with the quite large values of r_h and gives the increasing mistake if entropy becomes small.

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VI. APPENDIX: THE METHOD OF INTEGRATION OVER A PARAMETER OF ORDINARY DIFFERENTIAL EQUATIONS GIVEN IN A NON-EVIDENT FORM

Let us consider a system of usual differential equations with unknown functions $x_i(t) \in R^n$ which are not solved relatively derivatives $x'_i(t)$ to have the following form:

$$\begin{aligned} \sum_{j=1}^n x'_j(t) a_{1j}(x, t) &= b_1(x, t), \\ \sum_{j=1}^n x'_j(t) a_{2j}(x, t) &= b_2(x, t), \\ &\dots \\ \sum_{j=1}^n x'_j(t) a_{nj}(x, t) &= b_n(x, t), \end{aligned} \quad (35)$$

where the coefficients $b_i, a_{ij} \in C^1[R^n \times R^1]$. For the simplicity it is convenient to introduce $(n \times n)$ matrix $A(x, t) \equiv \{a_{ij}(x, t)\}_{i,j=1,\dots,n}$. The system (35) is a linear $x'_i(t)$ system, therefore, it can be solved and its solution has the following matrix form:

$$x'(t) = A^{-1}(x, t) b^\dagger(x, t), \quad (36)$$

where transposed vector $b^\dagger(x, t)$ is equal to $(b_1(x, t), b_2(x, t), \dots, b_n(x, t))^\dagger$. Let t_0 to be the unit value of t and $x(t_0) = x_0$. So, one has the Cauchy problem that is to continue the solution of (35) to the manifold $t \in [t_0, t_1]$. According to the existence theorem, the solution $x(t, t_0, x_0)$ of (35) would exist only in the case of the main system discriminant to be not equal to zero on the solutions of (35), beginning from the point (x_0, t_0) , i.e.:

$$\det A(x(t, t_0, x_0), t) \neq 0. \quad (37)$$

If in any point t^* the following condition is realized, i.e. $\det A(x(t, t_0, x_0), t) = 0$, the system (35) has a non-single solution, and a numerical integration performed by the usual Runge-Kutta, Adams, ets. method will stop before the point t^* . So, it is necessary to choose such a numerical integration strategy which allows to pass the point t^* . The same method used for other problem classes was discussed early, for example, by V.A.Egorov et all [13].

For the purposes discussed in the paper one can introduce a new independent variable s and further to consider x and t to be functions of s : $x = x(s)$, $t = t(s)$. Therefore,

$$\frac{dx}{dt} = \frac{dx}{ds} \frac{ds}{dt} = x'_s(t'_s)^{-1}. \quad (38)$$

Putting (38) to the system (35), one obtains a new homogeneous system which consists of n equations with $(n + 1)$ unknowns. One needs to add $(n + 1)$ th equation representing the normalization condition to the system, which takes the following form:

$$\begin{aligned}
\sum_{j=1}^n \frac{dx_j}{ds} a_{1j}(x, t) - \frac{dt}{ds} b_1(x, t) &= 0, \\
\sum_{j=1}^n \frac{dx_j}{ds} a_{2j}(x, t) - \frac{dt}{ds} b_2(x, t) &= 0, \\
&\dots \\
\sum_{j=1}^n \frac{dx_j}{ds} a_{nj}(x, t) - \frac{dt}{ds} b_n(x, t) &= 0, \\
\sum_{j=1}^n \frac{dx_j}{ds} \tau_j(s) - \frac{dt}{ds} \tau_t(s) &= 1,
\end{aligned} \tag{39}$$

where new functions $\tau(s) \in R^{n+1}$. These new functions $\tau(s)$ are fixed by the following normalization conditions:

$$\tau_i(s) = \frac{dx_i/ds}{\sqrt{\sum_{j=1}^n (dx_j/ds)^2 + (dt/ds)^2}}, \quad \tau_t(s) = \frac{dt/ds}{\sqrt{\sum_{j=1}^n (dx_j/ds)^2 + (dt/ds)^2}}, \tag{40}$$

So, $\vec{\tau}(s)$ is the tangent vector to the solution trajectory in $(n + 1)$ phase-space, therefore, the integration is proceeded along the solution trajectory, not along the radial coordinate r , as usual performed. At a first step the most convenient choice of $\vec{\tau}$ is the following: $\tau_i = 0$, $\tau_t = \pm 1$, where its sign depends upon the solution direction moving. At the next step $\vec{\tau}(s)$ are extrapolated by the Legendre polynoms using the $\vec{\tau}(s)$ values from the previous steps according to formulae (40). Therefore the system (39) remains the linear one. One can argue that if the integrating step vanishes the solution of the system (39) with the extrapolating functions $\vec{\tau}(s)$ reduce to the solution of the system (39) with the functions $\vec{\tau}(s)$ defined in (40). In such case our system (6), (8), (9) is written as follows:

$$\begin{aligned}
\frac{d\Delta}{ds} + 0 + 0 + 0 + 0 - \Delta' \frac{dr}{ds} &= 0, \\
0 + \frac{d\phi}{ds} + 0 + 0 + 0 - \phi' \frac{dr}{ds} &= 0, \\
0 + 0 + a_{11} \frac{d\Delta'}{ds} + a_{12} \frac{d\sigma}{ds} + a_{13} \frac{d\phi'}{ds} - b_1 \frac{dr}{ds} &= 0, \\
0 + 0 + a_{21} \frac{d\Delta'}{ds} + a_{22} \frac{d\sigma}{ds} + a_{23} \frac{d\phi'}{ds} - b_2 \frac{dr}{ds} &= 0, \\
0 + 0 + a_{31} \frac{d\Delta'}{ds} + a_{32} \frac{d\sigma}{ds} + a_{33} \frac{d\phi'}{ds} - b_3 \frac{dr}{ds} &= 0, \\
\frac{d\Delta}{ds} \tau_\Delta + \frac{d\phi}{ds} \tau_\phi + \frac{d\Delta'}{ds} \tau_{\Delta'} + \frac{d\sigma}{ds} \tau_\sigma + \frac{d\phi'}{ds} \tau_{\phi'} - \tau_r \frac{dr}{ds} &= 1,
\end{aligned} \tag{41}$$

where stroke denotes $\partial/\partial r$ and a_{ij} , b_i have the form:

$$\begin{aligned}
a_{11} &= 0, \\
a_{12} &= -m_{pl}^2 \sigma^2 r + 4e^{-2\phi} \lambda \phi' (\sigma^2 - 3\Delta), \\
a_{13} &= 4e^{-2\phi} \lambda \sigma (\Delta - \sigma^2), \\
a_{21} &= m_{pl}^2 \sigma^3 r + 4e^{-2\phi} \lambda \phi' 2\Delta \sigma, \\
a_{22} &= -m_{pl}^2 \sigma^2 (\Delta' r + 2\Delta) - 4e^{-2\phi} \lambda \phi' 2\Delta 3\Delta', \\
a_{23} &= 4e^{-2\phi} \lambda 2\Delta \Delta' \sigma, \\
a_{31} &= 4e^{-2\phi} \lambda (\Delta \sigma - \sigma^3), \\
a_{32} &= 2m_{pl}^2 \sigma^2 \Delta r^2 \phi' + 4e^{-2\phi} \lambda (-3\Delta \Delta' + \Delta' \sigma^2), \\
a_{33} &= -2m_{pl}^2 \sigma^2 \Delta r^2 \sigma,
\end{aligned}$$

$$\begin{aligned}
b_1 &= -m_{pl}^2 \sigma^3 r^2 (\phi')^2 + 4e^{-2\phi} \lambda \sigma (\Delta - \sigma^2) 2(\phi')^2, \\
b_2 &= -m_{pl}^2 \sigma^2 (2\Delta' \sigma + 2\Delta r \sigma (\phi')^2) + 4e^{-2\phi} \lambda 2\sigma \Delta \Delta' 2(\phi')^2 - 4e^{-2\phi} \lambda \phi' 2(\Delta')^2 \sigma, \\
b_3 &= 2m_{pl}^2 \sigma^2 (\Delta' r^2 \sigma \phi' + 2\Delta r \sigma \phi') - 4e^{-2\phi} \lambda (\Delta')^2 \sigma.
\end{aligned}$$

The integration is proceeded by the independent variable quantity s and starts from the point of $s = 0$. At the first step it is necessary the following condition to be fulfilled: $\det B = \det A \neq 0$, where B is the equation matrix defined as follows:

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -\Delta' \\ 0 & 1 & 0 & 0 & 0 & -\phi' \\ 0 & 0 & a_{11} & a_{12} & a_{13} & -b_1 \\ 0 & 0 & a_{21} & a_{22} & a_{23} & -b_2 \\ 0 & 0 & a_{31} & a_{32} & a_{33} & -b_3 \\ \tau_\Delta & \tau_\phi & \tau_{\Delta'} & \tau_\sigma & \tau_{\phi'} & \tau_r \end{pmatrix} \quad (42)$$

At every integration step one calculates $\bar{\tau}$ by the formulae (40). The 8th order Runge-Kutta program for integration by one step was used in our work. This program divides the given step automatically to reach the accuracy required.

One can argue that if the integration step becomes quite small, the matrix B (42) will not be degenerated at that trajectory point and $\text{rang} A \geq (n-1)$. When $\text{rang} A < (n-1)$, a ramifying point for two or more branches can exist in that point. Such a point can not be passed by the describing strategy.

Matrix B can be turned out by the Singular Value Decomposition (SVD) method [14]. It decomposes the given matrix B to the product of three matrixes $U\Sigma V^\dagger$, where U and V are the unitary matrixes $(n+1) \times (n+1)$ dimension and Σ is the diagonal matrix of absolute own values of B , i.e. $\Sigma = \{\sigma_{ii}\}$, $\sigma_{ii} \geq 0$. Therefore, $B^{-1} = V\Sigma^{-1}U^\dagger$ where $\Sigma^{-1} = \{1/\sigma_{ii}\}$. The most effective SVD algorithm can be found in the Numerical Analytical Group (NAG). It is very useful because during the calculation one sees the degeneracy's order of matrix B (singular values σ_{ii} vanish).

In the calculations it is desirable to work with Plank unit values, setting $m_{pl} = 1$. For simplicity the parameter λ in the action (1) was also set to be equal to one. Equations (20), (31) were added to the main system (41) to reach the maximum available accuracy for calculating ADM mass M , dilaton charge D and Euclidean action I_E .

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FIG. 1. The dependence of the metric functions Δ (a), σ (b) and dilaton function $\exp(-2\phi)$ (c) versus the radial coordinate r when the event horizon value r_h is equal to 10.0.

FIG. 2. 3D plots of the dependence of the metric function $\Delta(r)$ (a) and the dilaton function $\exp(-2\phi(r))$ (b) versus the event horizon value r_h .

FIG. 3. The entropy behavior of the Schwarzschild solution (line) and GB one (squares) versus the event horizon value r_h .

TABLE I. The dependence of ADM mass M , dilaton charge D , GB inverse temperature β and Schwarzschild one β_{SW} versus the event horizon value r_h .

r_h	3.5	5.0	10.0	15.0	20.0	30.0	100.0
M	1.8531	2.5365	5.0048	7.5014	10.0006	15.0001	50.0000
D	-0.4658	-0.3632	-0.1951	-0.1319	-0.0994	-0.0665	-0.01999
β	43.1675	62.8607	125.6630	188.4898	251.2456	376.9911	1256.6370
β_{SW}	43.9824	62.8320	125.6640	188.4960	251.3280	376.9920	1256.6400